

INTERPOLATING LOCAL CONSTANTS IN FAMILIES

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1. INTRODUCTION

Let $G = GL_n(F)$, let k be an algebraically closed field of characteristic ℓ , and $W(k)$ its ring of Witt vectors. By an ℓ -adic family of representations we mean an $A[G]$ -module V where A is a commutative $W(k)$ -algebra with unit; then each point \mathfrak{p} of A gives a $\kappa(\mathfrak{p})[G]$ -module $V \otimes_A \kappa(\mathfrak{p})$ where $\kappa(\mathfrak{p})$ denotes the residue field at \mathfrak{p} . In [EH12], Emerton and Helm conjecture a local Langlands correspondence for ℓ -adic families of admissible representations. To any continuous Galois representation $\rho : G_F \rightarrow GL_n(A)$, they conjecturally associate an admissible smooth $A[G]$ -module $\pi(\rho)$, which interpolates the local Langlands correspondence for points $A \rightarrow \kappa$ with κ characteristic zero. They prove that any $A[G]$ -module which is subject to this interpolation property and a short list of representation-theoretic conditions (see [EH12, Thm 6.2.1]) must be unique.

In [Hel12b], Helm further investigates the structure of $\pi(\rho)$ by taking the list of representation-theoretic conditions in [EH12, Thm 6.2.1] as a starting point for the theory of “co-Whittaker” $A[G]$ -modules (see Section 2.5 below for the definitions). Using this theory, he is able to reformulate the conjecture in terms of the existence of a certain homomorphism between the integral Bernstein center and a universal deformation ring ([Hel12b, Thm 7.8]).

Roughly speaking, representations of $GL_n(F)$ over \mathbb{C} are completely determined by data involving only local constants ([Hen93]), and in particular the bijections of the classical local Langlands correspondence are uniquely determined using L - and epsilon-factors (see, for example, [Jia13]). However, L - and epsilon-factors are absent from the local Langlands correspondence in families. Thus it is natural to ask whether it is possible to attach L - and epsilon-factors to an ℓ -adic family such as $\pi(\rho)$ as in [EH12], or more generally any co-Whittaker $A[G]$ -module, in a way that interpolates the L - and epsilon factors at each point.

Over \mathbb{C} , L -factors $L(\pi, X)$ arise as the greatest common denominator of the zeta integrals $Z(W, X; j)$ of a representation π as W varies over the space $\mathcal{W}(\pi, \psi)$ of Whittaker functions (see Sections 2.2, 3.1 for definitions). Epsilon-factors $\epsilon(\pi, X, \psi)$ are the constant of proportionality (i.e. not depending on W) in a functional equation relating the modified zeta integral $\frac{Z(W, X)}{L(\pi, X)}$ to its pre-composition with a Fourier transform. Here, the formal variable X replaces the complex variable $q^{-(s + \frac{n-1}{2})}$ appearing in [JPSS79] and other literature, and we consider these objects as formal series.

It appears difficult to construct L -factors in a way compatible with arbitrary change of coefficients. To see this, consider the following simple example: let $q \equiv 1 \pmod{\ell}$, and let $\chi_1, \chi_2 : F^\times \rightarrow W(k)^\times$ be smooth characters such that χ_1 is

unramified but χ_2 is ramified, and such that $\chi_1 \equiv \chi_2 \pmod{\ell}$. Following the classical procedure (see for example [BH06, 23.2]) for finding a generator of the fractional ideal of zeta integrals, we get $L(\chi_i, X) \in W(k)(X)$ and find that $L(\chi_1, X) = \frac{1}{1 - \chi_1(\varpi_F)X}$, and $L(\chi_2, X) = 1$. Now let A be the Noetherian local ring $\{(a, b) \in W(k) \times W(k) : a \equiv b \pmod{\ell}\}$, which has two characteristic zero points $\mathfrak{p}_1, \mathfrak{p}_2$ and a maximal ideal ℓA . Let π be the $A[F^\times]$ -module A , with the action of F^\times given by $x \cdot (a, b) = (\chi_1(x)a, \chi_2(x)b)$. Interpolating $L(\chi_1, X)$ and $L(\chi_2, X)$ would mean finding an element $L(\pi, X)$ in $A[[X]][X^{-1}]$ such that $L(\pi, X) \equiv L(\chi_i, X) \pmod{\ell}$ for $i = 1, 2$, but such a task is impossible because $L(\chi_1, X)$ and $L(\chi_2, X)$ are different mod ℓ .

On the other hand, zeta integrals themselves seem to be much more well-behaved with respect to specialization. Classically, zeta integrals form elements of the quotient field $\mathbb{C}(X)$ of $\mathbb{C}[X, X^{-1}]$. Our first result is identifying, for more arbitrary coefficient rings A , the correct fraction ring in which our naive generalization of zeta factors will live:

Theorem 1.1. *Suppose A is a Noetherian $W(k)$ -algebra. Let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then if V is a co-Whittaker $A[G]$ -module, $Z(W, X; j)$ lies in the fraction ring $S^{-1}(A[X, X^{-1}])$ for all $W \in \mathcal{W}(V, \psi)$ and for $0 \leq j \leq n - 2$.*

The proof of rationality in the setting of representations over a field relies on a useful decomposition of a Whittaker function into “finite” functions ([JPSS79, Prop 2.2]). In the setting of rings, such a structure theorem is lacking, but certain elements of its proof can be translated into a question about the finiteness of the $(n - 1)$ st Bernstein-Zelevinsky derivative. This finiteness property, combined with a simple translation property of the zeta integrals, yields Theorem 1.1 (see §3.2).

Classically, zeta integrals satisfy a functional equation which does not involve dividing by the L -factor. The constant of proportionality in this functional equation is called the gamma-factor and equals $\epsilon(\pi, X, \psi) \frac{L(\pi^\vee, \frac{1}{q^n X})}{L(\pi, X)}$, when the L -factor makes sense. Our second main result is that gamma-factors interpolate in ℓ -adic families (see §4.1 for details on the notation):

Theorem 1.2. *Suppose A is a Noetherian $W(k)$ -algebra and suppose V is a primitive co-Whittaker $A[G]$ -module. Then there exists a unique element $\gamma(V, X, \psi)$ of $S^{-1}(A[X, X^{-1}])$ such that*

$$Z(W, X; j) \gamma(V, X, \psi) = Z(\widetilde{w'W}, \frac{1}{q^n X}; n - 2 - j)$$

for any $W \in \mathcal{W}(V, \psi)$ and for any $0 \leq j \leq n - 2$.

To prove Theorem 1.2 we use the theory of the integral Bernstein center to reduce to the characteristic zero case of [JPSS79].

The question of interpolating local constants in ℓ -adic families has been investigated in a simple case by Vigneras in [Vig00]. For supercuspidal representations of $GL_2(F)$ over $\overline{\mathbb{Q}_\ell}$, Vigneras notes in [Vig00] that it is known that epsilon factors define elements of $\overline{\mathbb{Z}_\ell}$, and proves that for two supercuspidal integral representations to be congruent modulo ℓ it is necessary and sufficient that they have epsilon factors which are congruent modulo ℓ (we call a representation with coefficients in a local field E integral if it stabilizes an \mathcal{O}_E -lattice). The classical epsilon and gamma factors are equal in the supercuspidal case, so when the specialization of an

ℓ -adic family at a characteristic zero point is supercuspidal, the gamma factor we construct in this paper specializes to the epsilon factor of [JPSS79, Vig00]. Since two representations V_1, V_2 over \mathcal{O}_E which are congruent mod \mathfrak{m}_E define a family $V_1 \times_{\nabla} V_2$ over the connected $W(k)$ -algebra $\mathcal{O}_E \times_{k_E} \mathcal{O}_E$, Theorems 1.1 and 1.2 give the following corollary (implying the “necessary” part of [Vig00]):

Corollary 1.3. *Let K denote the fraction field of $W(k)$. If π and π' are absolutely irreducible integral representations of $GL_n(F)$ over a coefficient field E which is a finite extension of K , then:*

- (1) $\gamma(\pi, X, \psi)$ and $\gamma(\pi', X, \psi)$ have coefficients in the fraction ring $S^{-1}(\mathcal{O}_E[X, X^{-1}])$.
- (2) If \mathfrak{m}_E is the maximal ideal of \mathcal{O}_E , and $\pi \equiv \pi' \pmod{\mathfrak{m}_E}$, then $\gamma(\pi, X, \psi) \equiv \gamma(\pi', X, \psi) \pmod{\mathfrak{m}_E}$.

The question of extending the theory of zeta integrals to the ℓ -modular setting has been investigated in [M12], and very recently in [KM14] for the Rankin-Selberg integrals. The question of deforming local constants over polynomial rings over \mathbb{C} has been investigated by Cogdell and Piatetski-Shapiro in [CPS10], and the techniques of this paper owe much to those in [CPS10].

Analogous to the results of Bernstein and Deligne in [BD84] for $\text{Rep}_{\mathbb{C}}(G)$, Helm shows in [Hel12a, Thm 10.8] that the category $\text{Rep}_{W(k)}(G)$ has a decomposition into full subcategories known as blocks. Our third main result is constructing for each block a gamma factor which is universal in the sense that it gives rise via specialization to the gamma factor for any co-Whittaker module in that block. We will now state this result more precisely.

Each block of the category $\text{Rep}_{W(k)}(G)$ corresponds to a primitive idempotent in the Bernstein center \mathcal{Z} , which is defined as the ring of endomorphisms of the identity functor. It is a commutative ring whose elements consist of collections of compatible endomorphisms of every object, each such endomorphism commuting with all morphisms. Choosing a primitive idempotent e of \mathcal{Z} , the ring $e\mathcal{Z}$ is the center of the subcategory $e \cdot \text{Rep}_{W(k)}(G)$ of representations satisfying $eV = V$. The ring $e\mathcal{Z}$ has an interpretation as the ring of regular functions on an affine algebraic variety over $W(k)$, whose k -points are in bijection with the set of unramified twists of a fixed conjugacy class of supercuspidal supports in $\text{Rep}_k(G)$. See [Hel12a] for details. In [Hel12b], Helm determines a “universal co-Whittaker module” with coefficients in $e\mathcal{Z}$, denoted here by $e\mathfrak{W}$, which gives rise to any co-Whittaker module via specialization (see Proposition 2.31 below). By applying our theory of zeta integrals to $e\mathfrak{W}$ we get a gamma factor which is universal in the same sense:

Theorem 1.4. *Suppose A is any Noetherian $W(k)$ -algebra, and suppose V is a primitive co-Whittaker $A[G]$ -module. Then there is a primitive idempotent e , a homomorphism $f_V : e\mathcal{Z} \rightarrow A$, and an element $\Gamma(e\mathfrak{W}, X, \psi) \in S^{-1}(e\mathcal{Z}[X, X^{-1}])$ such that $\gamma(V, X, \psi) = f_V(\Gamma(e\mathfrak{W}, X, \psi))$.*

Interpolating gamma factors of pairs may be the next step in obtaining a local converse theorem for ℓ -adic families. By capturing the interpolation property, families of gamma factors might give an alternative characterization of the co-Whittaker module $\pi(\rho)$ appearing in the local Langlands correspondence in families.

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school on New Geometric Techniques in Number Theory in 2013. He would also like to thank the referee for her/his very helpful comments and suggestions.

1.1. Notation and Conventions. Let F be a finite extension of \mathbb{Q}_p , let q be the order of its residue field, and let k be an algebraically closed field of characteristic ℓ , where $\ell \neq p$ is an odd prime. Denote by $W(k)$ the ring of Witt vectors over k . The assumption that ℓ is odd is made so that $W(k)$ contains a square root of q . When $\ell = 2$ all the arguments presented will remain valid, after possibly adjoining a square root of q to $W(k)$. The letter G or G_n will always denote the group $GL_n(F)$. Throughout the paper A will be a Noetherian commutative ring which is a $W(k)$ -algebra, with additional properties in various sections, and $\kappa(\mathfrak{p})$ will denote the residue field of a prime ideal \mathfrak{p} of A . For any locally profinite group H , $\text{Rep}_A(H)$ denotes the category of smooth representations of H over the ring A , i.e. $A[H]$ -modules for which every element is stabilized by an open subgroup of H . Even when this category is not mentioned, all representations are presumed to be smooth. When H is a closed subgroup of G , we define the non-normalized induction functor Ind_H^G (resp. c-Ind_H^G) : $\text{Rep}_A(H) \rightarrow \text{Rep}_A(G)$ sending τ to the smooth part of the $A[G]$ -module, under right translation, of functions (resp. functions compactly supported modulo H) $f : G \rightarrow \tau$ such that $f(hg) = \tau(h)f(g)$, $h \in H$, $g \in G$.

The integral Bernstein center of [Hel12a] (see the discussion preceding Theorem 1.4) will always be denoted by \mathcal{Z} . If V is in $\text{Rep}_A(G)$, then it is also in $\text{Rep}_{W(k)}(G)$, and we frequently use the Bernstein decomposition of $\text{Rep}_{W(k)}(G)$ to interpret properties of V .

If A has a nontrivial ideal I , then $I \cdot V$ is an $A[H]$ -submodule of V , which shows that most content would be missing if we developed the representation theory of $\text{Rep}_A(H)$ around the notion of irreducible objects, or simple $A[H]$ -modules. Thus conditions appear throughout the paper which in the traditional setting are implied by irreducibility:

Definition 1.5. V in $\text{Rep}_A(H)$ will be called

- (1) Schur if the natural map $A \rightarrow \text{End}_{A[G]}(V)$ is an isomorphism;
- (2) G -finite if it is finitely generated as an $A[G]$ -module.
- (3) primitive if there exists a primitive idempotent e in the Bernstein center \mathcal{Z} such that $eV = V$.

We say a ring is connected if it has connected spectrum or, equivalently, no nontrivial idempotents, for example any local ring or integral domain. Note that if A is connected, Corollary 2.32 implies all co-Whittaker $A[G]$ -modules are primitive.

Denote by N_n the subgroup of G_n consisting of all unipotent upper-triangular matrices. Let $\psi : F \rightarrow W(k)^\times$ be an additive character of F with $\ker \psi = \mathfrak{p}$. Then ψ defines a character on any subgroup of $N_n(F)$ by

$$(u)_{i,j} \mapsto \psi(u_{1,2} + \cdots + u_{n-1,n});$$

we abusively denote this character by ψ as well.

If H is a subgroup normalized by another subgroup group K , and θ is a character of the group H , denote by θ^k the character given by $\theta^k(h) = \theta(khk^{-1})$ for $h \in H$, $k \in K$. For V in $\text{Rep}_A(H)$, denote by $V_{H,\theta}$ the quotient $V/V(H,\theta)$ where $V(H,\theta)$ is the sub- A -module generated by elements of the form $hv - \theta(h)v$ for $h \in H$ and $v \in V$; it is K -stable if $\theta^k = \theta$, $k \in K$. Given a standard Levi subgroup $M \subset G_n$

with unipotent radical U , and $\mathbf{1}$ the trivial character, we denote by J_M the non-normalized Jacquet functor $\text{Rep}_A(G) \rightarrow \text{Rep}_A(M) : V \mapsto V_{U, \mathbf{1}}$.

For each $m \leq n$, let G_m denote $GL_m(F)$ and embed it in G via $\begin{pmatrix} G_m & 0 \\ 0 & I_{n-m} \end{pmatrix}$. We let $\{1\} = P_1 \subset \cdots \subset P_n$ denote the mirabolic subgroups of $G_1 \subset \cdots \subset G_n$, where P_m is given by $\{ \begin{pmatrix} g_{m-1} & x \\ 0 & 1 \end{pmatrix} : g_{m-1} \in G_{m-1}, x \in F^{m-1} \}$. We also have the unipotent upper triangular subgroup U_m of P_m given by $\{ \begin{pmatrix} I_{m-1} & x \\ 0 & 1 \end{pmatrix} : x \in F^{m-1} \}$. In particular, $U_m \simeq F^{m-1}$ and $P_m = U_m G_{m-1}$. Note that this is different from the groups $N(r)$ defined in Proposition 2.3.

Since G_n contains a compact open subgroup whose pro-order is invertible in $W(k)$, there exists a unique (for that choice of subgroup) normalized Haar measure, defining integration on the space $C_c^\infty(G, A)$ of smooth compactly supported functions $G \rightarrow A$ ([Vig96, I.2.3]).

2. REPRESENTATION THEORETIC BACKGROUND

2.1. Co-invariants and Derivatives. As in [EH12, BZ77], we define the following functors with respect to the character ψ .

$$\begin{aligned} \Psi^+ : \text{Rep}_A(P_{n-1}) &\rightarrow \text{Rep}_A(P_n) & \Psi^+ : \text{Rep}(G_{n-1}) &\rightarrow \text{Rep}(P_n) \\ V &\mapsto \text{c-Ind}_{P_{n-1}U_n}^{P_n} V \text{ (} U_n \text{ acts via } \psi \text{)} & V &\mapsto V \text{ (} U_n \text{ acts trivially)} \\ \hat{\Phi}^+ : \text{Rep}(P_{n-1}) &\rightarrow \text{Rep}(P_n) & \Psi^- : \text{Rep}(P_n) &\rightarrow \text{Rep}(G_{n-1}) \\ V &\mapsto \text{Ind}_{P_{n-1}U_n}^{P_n} V & V &\mapsto V/V(U_n, \mathbf{1}) \\ \Phi^- : \text{Rep}(P_n) &\rightarrow \text{Rep}(P_{n-1}) & & \\ V &\mapsto V/V(U_n, \psi) & & \end{aligned}$$

Note that we give these functors the same names as the ones originally defined in [BZ76], but we use the non-normalized induction functors, as in [BZ77, EH12], because they are simpler for our purposes. As observed in [EH12], these functors retain the basic adjointness properties proved in [BZ77, §3.2]. This is because the methods of proof in [BZ76, BZ77] use properties of l -sheaves which carry over to the setting of smooth $A[G]$ -modules where A is a Noetherian $W(k)$ -algebra.

Proposition 2.1 ([EH12], 3.1.3). (1) *The functors Ψ^- , Ψ^+ , Φ^- , Φ^+ , $\hat{\Phi}^+$ are exact.*

(2) *Φ^+ is left adjoint to Φ^- , Ψ^- is left adjoint to Ψ^+ , and Φ^- is left adjoint to $\hat{\Phi}^+$.*

(3) $\Psi^- \Phi^+ = \Phi^- \Psi^+ = 0$

(4) $\Psi^- \Psi^+$, $\Phi^- \hat{\Phi}^+$, and $\Phi^- \Phi^+$ are naturally isomorphic to the identity functor.

(5) *For each V in $\text{Rep}(P_n)$ we have an exact sequence*

$$0 \rightarrow \Phi^+ \Phi^-(V) \rightarrow V \rightarrow \Psi^+ \Psi^-(V) \rightarrow 0.$$

(6) *(Commutativity with Tensor Product) If M is an A -module and F is Ψ^- , Ψ^+ , Φ^- , Φ^+ , or $\hat{\Phi}^+$, we have $F(V \otimes_A M) \cong F(V) \otimes_A M$.*

For $1 \leq m \leq n$ we define the m th derivative functor

$$(-)^{(m)} := \Psi^-(\Phi^-)^{m-1} : \text{Rep}(P_n) \rightarrow \text{Rep}(G_{n-m}).$$

This gives a functor $\text{Rep}(G_n) \rightarrow \text{Rep}(G_{n-m})$ by first restricting representations to P_n and then applying $(-)^{(m)}$; this functor is also denoted $(-)^{(m)}$. The zero'th derivative functor $(-)^{(0)}$ is the identity. We can describe the derivative functor $(-)^{(m)}$ more explicitly by using the following lemma on the transitivity of coinvariants:

Lemma 2.2 ([BZ76] §2.32). *Let H be a locally profinite group, θ a character of H , and V a representation of H . Suppose H_1, H_2 are subgroups of H such that $H_1 H_2 = H$ and H_1 normalizes H_2 . Then*

$$\left(V_{H_2, \theta|_{H_2}} \right)_{H_1, \theta|_{H_1}} = V_{H, \theta}.$$

Define $N(r)$ to be the group of matrices whose first r columns are those of the identity matrix, and whose last $n - r$ columns are those of elements of N_n (recall that N_n is the group of unipotent upper triangular matrices). For $2 \leq r \leq n$ we have $U_r N(r) = N(r - 1)$ and U_r normalizes $N(r)$. As $N(r)$ is contained in N_n , we define ψ on $N(r)$ via its superdiagonal entries. We can also define a character $\tilde{\psi}$ on $N(r)$ slightly differently from the usual definition: $\tilde{\psi}$ will be given as usual via ψ on the last $n - r - 1$ superdiagonal entries, but trivially on the $(r, r + 1)$ entry, i.e.

$$\tilde{\psi}(x) := \psi(0 + x_{r+1, r+2} + \cdots + x_{n-1, n}) \text{ for } x \in N(r).$$

The functors $(\Phi^-)^m$ and $(-)^{(m)}$, defined above, can be described more explicitly. Let $m = n - r$. By applying Lemma 2.2 repeatedly with $H_1 = U_r$, and $H_2 = N(r - 1)$, we get

Proposition 2.3 ([Vig96] III.1.8). (1) $(\Phi^-)^m V$ equals the module of coinvariants $V/V(N(n - m), \psi)$.
 (2) $V^{(m)}$ equals the module of coinvariants $V/V(N(n - m), \tilde{\psi})$.

In particular, if $m = n$, this gives $V^{(n)} = V/V(N_n, \psi)$. Note that $V^{(n)}$ is simply an A -module.

2.2. Whittaker and Kirillov Functions. The character $\psi : N_n \rightarrow A^\times$ defines a representation of N_n in the A -module A , which we also denote by ψ . By Proposition 2.3 we have $\text{Hom}_A(V^{(n)}, A) = \text{Hom}_{N_n}(V, \psi)$.

Definition 2.4. For V in $\text{Rep}_A(G_n)$, we say that V is of Whittaker type if $V^{(n)}$ is free of rank one as an A -module. As in [EH12, Def 3.1.8], if A is a field we refer to representations of Whittaker type as generic.

If V is of Whittaker type, $\text{Hom}_{N_n}(V, \psi)$ is free of rank one, so we may choose a generator λ . The image of λ under the Frobenius reciprocity isomorphism $\text{Hom}_{N_n}(V, \psi) \xrightarrow{\sim} \text{Hom}_{G_n}(V, \text{Ind}_{N_n}^{G_n} \psi)$ is the map $v \mapsto W_v$ where $W_v(g) = \lambda(gv)$. The $A[G]$ -module formed by the image of the map $v \mapsto W_v$ is independent of the choice of λ .

Definition 2.5. The image of the homomorphism $V \rightarrow \text{Ind}_{N_n}^{G_n} \psi$ is called the space of Whittaker functions of V and is denoted $\mathcal{W}(V, \psi)$ or just \mathcal{W} .

Choosing a generator of $V^{(n)}$ and allowing N_n to act via ψ , we get an isomorphism $V^{(n)} \xrightarrow{\sim} \psi$. Composing this with the natural quotient map $V \rightarrow V^{(n)}$ gives an N_n -equivariant map $V \rightarrow \psi$, which is a generator λ . Note that the map $V \rightarrow \mathcal{W}(V, \psi)$ is surjective but not necessarily an isomorphism, unlike the setting of irreducible generic representations with coefficients in a field. Different $A[G]$ -modules of Whittaker type can have the same space of Whittaker functions:

Lemma 2.6. *Suppose V', V in $\text{Rep}_A(G)$ are of Whittaker type, and suppose there is a G -equivariant homomorphism $\alpha : V' \rightarrow V$ such that $\alpha^{(n)} : (V')^{(n)} \rightarrow V^{(n)}$ is an isomorphism. Then $\mathcal{W}(V', \psi)$ is the subrepresentation of $\mathcal{W}(V, \psi)$ given by $\mathcal{W}(\alpha(V'), \psi)$.*

Proof. Let $q' : V' \rightarrow V'/V'(N_n, \psi)$ and $q : V \rightarrow V/V(N_n, \psi)$ be the quotient maps. Choosing a generator for $V^{(n)}$ gives isomorphisms η, η' such that the following diagram commutes.

$$\begin{array}{ccccc} V & \xrightarrow{q} & V^{(n)} & \xrightarrow{\eta} & A \\ \alpha \downarrow & & \alpha^{(n)} \downarrow & \nearrow \eta' & \\ V' & \xrightarrow{q'} & (V')^{(n)} & & \end{array}$$

Given $v' \in V'$ we get

$$W_{\alpha(v')}(g) = \eta(q(g\alpha v')) = \eta((q \circ \alpha)(gv')) = \eta'(q'(gv')) = W_{v'}(g), \quad g \in G.$$

This shows $\mathcal{W}(V', \psi) = \mathcal{W}(\alpha(V'), \psi) \subset \mathcal{W}(V, \psi)$. \square

If V in $\text{Rep}_A(G_n)$ is Whittaker type and $v \in V$, we will denote by $W_v|_{P_n}$ the restriction of the function W_v to the subgroup $P_n \subset G_n$.

Definition 2.7. *The image of the P_n -equivariant homomorphism $V \rightarrow \text{Ind}_{N_n}^{P_n} \psi : v \mapsto W_v|_{P_n}$ is called the Kirillov functions of V and is denoted $\mathcal{K}(V, \psi)$ or just \mathcal{K} .*

The following properties of the Kirillov functions are well known for $\text{Rep}_{\mathbb{C}}(G)$, but we will need them for $\text{Rep}_A(G)$:

Proposition 2.8. *Let V be of Whittaker type in $\text{Rep}_A(P_n)$, and choose a generator of $V^{(n)}$ in order to identify $V^{(n)}$ with A . Then the following hold:*

- (1) $(\Phi^+)^{n-1}V^{(n)} = \text{c-Ind}_{N_n}^{P_n} \psi$ and $(\hat{\Phi}^+)^{n-1}V^{(n)} = \text{Ind}_{N_n}^{P_n} \psi$.
- (2) The composition $(\Phi^+)^{n-1}V^{(n)} \rightarrow V \rightarrow \text{Ind}_{N_n}^{P_n} \psi$ differs from the inclusion $\text{c-Ind}_{N_n}^{P_n} \psi \hookrightarrow \text{Ind}_{N_n}^{P_n} \psi$ by multiplication with an element of A^\times .
- (3) The Kirillov functions $\mathcal{K}(V, \psi)$ contains $\text{c-Ind}_{N_n}^{P_n} \psi$ as a sub- $A[P_n]$ -module.

Proof. The proof in [BZ76] Proposition 5.12 (g) works to prove (1) in this context.

Let $\mathfrak{S} = (\Phi^+)^{n-1}V^{(n)}$. There is an embedding $\mathfrak{S} \rightarrow V$ by Proposition 2.1 (5); denote by t the composition $\mathfrak{S} \rightarrow V \rightarrow \text{Ind} \psi$. Then $t^{(n)} : \mathfrak{S}^{(n)} \rightarrow \text{Ind} \psi^{(n)}$ is a nonzero homomorphism between free rank one A -modules, hence given by multiplication with an element a of A . By Proposition 2.1 (6), For any maximal ideal \mathfrak{m} of A , $t^{(n)} \otimes \kappa(\mathfrak{m})$ must be an isomorphism because it is a nonzero element of

$$\text{Hom}_{\kappa(\mathfrak{m})}((S(V) \otimes \kappa(\mathfrak{m}))^{(n)}, (\text{Ind} \psi \otimes \kappa(\mathfrak{m}))^{(n)}) = \kappa(\mathfrak{m}).$$

Thus a is nonzero in $\kappa(\mathfrak{m})$ for all \mathfrak{m} , hence a unit, so $t^{(n)}$ is an isomorphism. On the other hand there is the natural embedding $\text{c-Ind} \psi \rightarrow \text{Ind} \psi$, which we will denote s . Since $s^{(n)}$ is an isomorphism by [BZ77, Prop 3.2 (f)], we have $s^{(n)} = ut^{(n)}$ for some $u \in A^\times$. Thus, if $K := \ker(s - ut)$ then $K^{(n)} = S(V)^{(n)} = V^{(n)}$, whence $\text{Hom}_P(S(V)/K, \text{Ind} \psi) \cong \text{Hom}_A((S(V)/K)^{(n)}, A) = \text{Hom}_A(\{0\}, A) = 0$, which implies $s - ut \equiv 0$.

To prove (3), note that since $\mathcal{K}(V, \psi)$ is defined to be the image of the map $V \rightarrow \text{Ind}_{N_n}^{P_n} \psi$, this follows from (2). \square

Definition 2.9 ([EH12], §3.1). *If V is in $\text{Rep}(P_n)$, the image of the natural inclusion $(\Phi^+)^{n-1}V^{(n)} \rightarrow V$ is called the Schwartz functions of V and is denoted $\mathcal{S}(V)$. For V in $\text{Rep}(G_n)$ we also denote by $\mathcal{S}(V)$ the Schwartz functions of V restricted to P_n .*

We can ask how the functor Φ^- is reflected in the Kirillov space of a representation. First we observe that Φ^- commutes with the functor \mathcal{K} :

Lemma 2.10. *For $0 \leq m \leq n$, the $A[P_m]$ -modules $\mathcal{K}((\Phi^-)^{n-m}V, \psi)$ and $(\Phi^-)^{n-m}\mathcal{K}(V, \psi)$ are identical.*

Proof. The image of the P_{n-m} -submodule $V(N(m), \psi)$ in the map $V \rightarrow \mathcal{K}$ equals the submodule $\mathcal{K}(N(m), \psi)$. The lemma then follows from Proposition 2.3 \square

Following [CPS10], we can explicitly describe the effect of the functor Φ^- on the Kirillov functions \mathcal{K} . Recall that $\mathcal{K}(U_n, \psi)$ denotes the A -submodule generated by $\{uW - \psi(u)W : u \in U_n, W \in \mathcal{K}\}$ and $\Phi^- \mathcal{K} := \mathcal{K}/\mathcal{K}(U_n, \psi)$.

Proposition 2.11 ([CPS10] Prop 1.1).

$$\mathcal{K}(U_n, \psi) = \{W \in \mathcal{K} : W \equiv 0 \text{ on the subgroup } P_{n-1} \subset P_n\}.$$

Proof. The proof of [CPS10, Prop 1.1] carries over in this setting. It utilizes the Jacquet-Langlands criterion for an element v of a representation V to be in the subspace $V(U_{n_i}, \psi)$, which remains valid over more general coefficient rings A because all integrals are finite sums. \square

Thus Φ^- has the same effect as restriction of functions to the subgroup P_{n-1} inside P_n :

$$\Phi^- \mathcal{K} \cong \left\{ W \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} : W \in \mathcal{W}(V, \psi), p \in P_{n-1} \right\}.$$

By applying for each $m = 1, \dots, n-2$ the argument of [CPS10, Prop 1.1] to the P_{n-m+1} representation

$$\left\{ W \begin{pmatrix} p & 0 \\ 0 & I_{m-1} \end{pmatrix} : W \in \mathcal{W}(V, \psi), p \in P_{n-m+1} \right\}$$

instead of to \mathcal{K} , we can describe $(\Phi^-)^m \mathcal{K}$:

Corollary 2.12. *For $m = 1, \dots, n-1$,*

$$(\Phi^-)^m \mathcal{K} \cong \left\{ W \begin{pmatrix} p & 0 \\ 0 & I_m \end{pmatrix} : W \in \mathcal{W}(V, \psi), p \in P_{n-m} \right\}.$$

2.3. Partial Derivatives. Given a product $H_1 \times H_2$ of subgroups of G , and a character ψ on the unipotent upper triangular elements of H_2 , we can define “partial” versions of the functors Φ^\pm, Ψ^\pm as follows: given V in $\text{Rep}_A(H_1 \times H_2)$, restrict it to a representation of $H_1 = \{1\} \times H_2$, then apply the functor Φ^\pm or Ψ^\pm , and observe that $H_1 \times \{1\}$ acts naturally on the result, since it commutes with $\{1\} \times H_2$.

More precisely:

$$\begin{aligned}
\Phi^{+,2} : \text{Rep}_A(G_{n-m} \times P_{m-1}) &\rightarrow \text{Rep}_A(G_{n-m} \times P_m) \\
V &\mapsto \text{c-Ind}_{G_{n-m} \times P_{m-1} U_m}^{G_{n-m} \times P_m}(V), \quad \text{with } \{1\} \times U_m \text{ acting via } \psi \\
\hat{\Phi}^{+,2} : \text{Rep}(G_{n-m} \times P_{m-1}) &\rightarrow \text{Rep}(G_{n-m} \times P_m) \\
V &\mapsto \text{c-Ind}_{G_{n-m} \times P_{m-1} U_m}^{G_{n-m} \times P_m}(V) \\
\Phi^{-,2} : \text{Rep}(G_{n-m} \times P_m) &\rightarrow \text{Rep}(G_{n-m} \times P_{m-1}) \\
V &\mapsto V/V(\{1\} \times U_m, \psi) \\
\Psi^{+,2} : \text{Rep}(G_{n-m} \times G_{m-1}) &\rightarrow \text{Rep}(G_{n-m} \times P_m) \\
V &\mapsto V \quad (\{1\} \times U_m \text{ acts trivially}) \\
\Psi^{-,2} : \text{Rep}(G_{n-m} \times P_m) &\rightarrow \text{Rep}(G_{n-m} \times G_{m-1}) \\
V &\mapsto V/V(\{1\} \times U_m, \mathbf{1})
\end{aligned}$$

Because $H_1 \times \{1\}$ commutes with $\{1\} \times H_2$, we immediately get

Lemma 2.13. *The analogue of Proposition 2.1 (1)-(6) holds for $\Phi^{+,2}$, $\hat{\Phi}^{+,2}$, $\Phi^{-,2}$, $\Psi^{+,2}$, and $\Psi^{-,2}$.*

Definition 2.14. *We define the functor $(-)^{(0,m)} : \text{Rep}_A(G_{n-m} \times G_m) \rightarrow \text{Rep}_A(G_{n-m})$ to be the composition $\Psi^{-,2}(\Phi^{-,2})^{m-1}$.*

The proof of the following Proposition holds for $W(k)$ -algebras A :

Proposition 2.15 ([Zel80] Prop 6.7, [Vig96] III.1.8). *Let $M = G_{n-m} \times G_m$. For $0 \leq m \leq n$ the m 'th derivative functor $(-)^{(m)}$ is the composition of the Jacquet functor $J_M : \text{Rep}(G_n) \rightarrow \text{Rep}(G_{n-m} \times G_m)$ with the functor $(-)^{(0,m)} : \text{Rep}_A(G_{n-m} \times G_m) \rightarrow \text{Rep}_A(G_{n-m})$.*

Lemma 2.16. *Let V be in $\text{Rep}_A(G_{n-m} \times G_m)$. Then V contains an A -submodule isomorphic to $V^{(0,m)}$.*

Proof. The image of the natural embedding $(\Phi^{+,2})^{m-1} \Psi^{+,2}(V^{(0,m)}) \rightarrow V$, which is given by Proposition 2.13 (5), will be denoted $\mathcal{S}^{0,2}(V)$. By Proposition 2.13 (4), the natural surjection $V \rightarrow V^{(0,m)}$ restricts to a surjection $\mathcal{S}^{0,2}(V) \rightarrow V^{(0,m)}$. By Proposition 2.13 (6), the map of A -modules $\mathcal{S}^{0,2}(V) \rightarrow V^{(0,m)}$ arises from the map $(\Phi^{+,2})^{m-1} \Psi^{+,2}(A) \rightarrow A$ by tensoring over A with $V^{(0,m)}$. Take the A -submodule generated by any element of $(\Phi^{+,2})^{m-1} \Psi^{+,2}(A)$ that maps to the identity in A ; then tensor with $V^{(0,m)}$. \square

2.4. Finiteness Results. In this subsection we gather certain finiteness results involving derivatives, most of which are well-known when A is a field of characteristic zero.

Let H be any topological group containing a decreasing sequence $\{H_i\}_{i \geq 0}$ of open subgroups whose pro-order is invertible in A , and which forms a neighborhood base of the identity in H . If V is a smooth $A[H]$ -module we may define a projection $\pi_i : V \rightarrow V^{H_i} : v \mapsto \int_{H_i} h v$ for a Haar measure on H_i where H_i has total measure 1. The A -submodules $V_i := \ker(\pi_i) \cap V^{H_{i+1}}$ then satisfy $\bigoplus_i V_i = V$.

Lemma 2.17 ([EH12] Lemma 2.1.5, 2.1.6). *A smooth $A[H]$ -module V is admissible if and only if each A -module V_i is finitely generated. In particular, quotients of admissible $A[H]$ -modules by $A[H]$ -submodules are admissible.*

Thus the following version of the Nakayama lemma applies to admissible $A[H]$ -modules:

Lemma 2.18 ([EH12] Lemma 3.1.9). *Let A be a Noetherian local ring with maximal ideal \mathfrak{m} , and suppose that M is a submodule of a direct sum of finitely generated A -modules. If $M/\mathfrak{m}M$ is finite dimensional then M is finitely generated over A .*

If V is admissible, then it is G -finite if and only if $V/\mathfrak{m}V$ is G -finite. To see this, take $S \subset V/\mathfrak{m}V$ an $(A/\mathfrak{m})[H]$ -generating set, let W be the $A[H]$ -span of a lift to V . Since V/W is admissible, we can apply Nakayama to each factor $(V/W)_i$ to conclude $V/W = 0$.

Proposition 2.19 ([EH12] 3.1.7). *Let κ be a $W(k)$ -algebra which is a field, and V an absolutely irreducible admissible representation of G_n . Then $V^{(n)}$ is zero or one-dimensional over κ , and is one-dimensional if and only if V is cuspidal.*

Proposition 2.20 ([Vig96] II.5.10(b)). *Let κ be a $W(k)$ -algebra which is a field. If V is a $\kappa[G]$ -module, then V is admissible and G -finite if and only if V is finite length over $\kappa[G]$.*

Proof. Suppose V is admissible and G -finite. If κ were algebraically closed of characteristic zero (resp. characteristic ℓ), this is [BZ77, 4.1] (resp. [Vig96, II.5.10(b)]). Otherwise, let $\bar{\kappa}$ be an algebraic closure, then $V \otimes \bar{\kappa}$ is finite length, so V is finite length.

If V is finite length, so is $V \otimes_{\kappa} \bar{\kappa}$. Over an algebraically closed field of characteristic different from p , irreducible representations are admissible ([BZ77, 3.25], [Vig96, II.2.8]). Since admissibility is preserved under taking extensions $V \otimes \bar{\kappa}$ being finite length implies it is admissible, hence V is admissible. Thus we can reduce proving G -finiteness to proving that, given any exact sequence of admissible objects, $0 \rightarrow W_0 \rightarrow V \rightarrow W_1 \rightarrow 0$ where W_0 and W_1 are G -finite, then V is G -finite. But there is a compact open subgroup U such that W_0 and W_1 are generated by W_0^U and W_1^U , respectively. It follows that V is generated by V^U . \square

Lemma 2.21. *Let κ be a $W(k)$ -algebra which is a field. If V is an absolutely irreducible $\kappa[G_n]$ -module, then for $m \geq 0$, $V^{(m)}$ is finite length as a $\kappa[G_{n-m}]$ -module.*

Proof. We follow [Vig96, III.1.10]. Given j, k positive integers, let $M = G_j \times G_k$ and let $P = MN$ be the associated standard parabolic subgroup. Given τ in $\text{Rep}_{\kappa}(G_j)$ and σ in $\text{Rep}_{\kappa}(G_k)$, we define $\tau \times \sigma$ to be the normalized induction $\text{c-Ind}_P(\delta_N^{1/2}(\sigma \otimes \tau))$ in $\text{Rep}_{\kappa}(G_{j+k})$, where δ_N denotes the modulus character of N (for the definition of δ_N see [BZ77, 1.7]). There exists a multiset $\{\pi_1, \dots, \pi_r\}$ of irreducible cuspidals such that $V \subset \pi_1 \times \dots \times \pi_r$. The Liebniz formula for derivatives says that $(\pi_1 \times \pi_2)^{(t)}$ has a filtration whose successive quotients are $\pi_1^{(t-i)} \times \pi_2^{(i)}$. Its proof, given in [BZ77, §7], carries over in this generality. Then $V^{(m)} \subset (\pi_1 \times \dots \times \pi_r)^{(m)}$, which is finite length by induction, using Proposition 2.19 combined with the Liebniz formula. \square

Proposition 2.22 ([Hel12b] Prop 9.15). *Let M be a standard Levi subgroup of G . If V in $\text{Rep}_A(G)$ is admissible and primitive, then $J_M V$ in $\text{Rep}_A(M)$ is admissible.*

Corollary 2.23. *If A is a local Noetherian $W(k)$ -algebra and V is admissible and G -finite, then $V^{(m)}$ is admissible and G -finite for $0 \leq m \leq n$.*

Proof. Let $M = G_{n-m} \times G_m$. By Proposition 2.15, $V^{(m)} = (J_M V)^{(0,m)}$, so by Lemma 2.16, there is an embedding $V^{(m)} \rightarrow J_M V$. Admissibility and G -finiteness mean V is generated over $A[G]$ by vectors in V^K for some compact open subgroup K . Since V^K is finite over A , eV^K is nonzero for only a finite set of primitive idempotents e of the Bernstein center, so $eV \neq 0$ for at most finitely many primitive idempotents e of the integral Bernstein center. Therefore, Proposition 2.22 applies, showing $V^{(m)}$ embeds in an admissible module. Thus by Lemma 2.18, we are reduced to proving the result for $\overline{V} := V/\mathfrak{m}V$. Since \overline{V} is admissible and G -finite, and A/\mathfrak{m} is characteristic ℓ , Lemma 2.20 shows \overline{V} is finite length, therefore it follows from Lemma 2.21 that $\overline{V}^{(m)}$ is finite length. Applying Lemma 2.20 once more, we have the result. \square

Loosely speaking, the $(n-1)$ st derivative describes the restriction of a G_n -representation to a G_1 -representation (see Corollary 2.12). The next result shows that this restriction intertwines a finite set of characters:

Theorem 2.24. *If A is a local $W(k)$ -algebra and V in $\text{Rep}_A(G)$ is admissible and G -finite, then $V^{(n-1)}$ is finitely generated as an A -module.*

Proof. By Lemma 2.18 and Corollary 2.23 it is sufficient to show that $\overline{V}^{(n-1)}$ is finite over the residue field κ . We know $\overline{V}^{(n-1)}$ is G -finite and admissible by Corollary 2.23, hence finite length as a $\kappa[G_1]$ -module by Proposition 2.20. Since G_1 is abelian, all composition factors are 1-dimensional, so $\overline{V}^{(n-1)}$ being finite length implies it is finite dimensional over κ . \square

Since the hypotheses of being admissible and G -finite are preserved under localization by Proposition 2.1 (6), we can go beyond the local situation:

Corollary 2.25. *Let A be a Noetherian $W(k)$ -algebra and suppose that V is admissible and G -finite. Then for every \mathfrak{p} in $\text{Spec } A$, $V_{\mathfrak{p}}^{(n-1)}$ is finitely generated as an $A_{\mathfrak{p}}$ -module.*

2.5. Co-Whittaker $A[G]$ -Modules. In this subsection we define co-Whittaker representations and show that every admissible $A[G]$ -module V of Whittaker type contains a canonical co-Whittaker subrepresentation.

Definition 2.26 ([Hel12b] 3.3). *Let κ be a field of characteristic different from p . An admissible smooth object U in $\text{Rep}_{\kappa}(G)$ is said to have essentially AIG dual if it is finite length as a $\kappa[G]$ -module, its cosocle $\text{cos}(U)$ is absolutely irreducible generic, and $\text{cos}(U)^{(n)} = U^{(n)}$ (the cosocle of a module is its largest semisimple quotient).*

This condition is equivalent to $U^{(n)}$ being one-dimensional over κ and having the property that $W^{(n)} \neq 0$ for any nonzero quotient $\kappa[G]$ -module W (see [EH12, Lemma 6.3.5] for details).

Definition 2.27 ([Hel12b] 6.1). *An object V in $\text{Rep}_A(G)$ is said to be co-Whittaker if it is admissible, of Whittaker type, and $V \otimes_A \kappa(\mathfrak{p})$ has essentially AIG dual for each \mathfrak{p} .*

Proposition 2.28 ([Hel12b] Prop 6.2). *Let V be a co-Whittaker $A[G]$ -module. Then the natural map $A \rightarrow \text{End}_{A[G]}(V)$ is an isomorphism.*

Lemma 2.29. *Suppose V is admissible of Whittaker type and, for all primes \mathfrak{p} , any non-generic quotient of $V \otimes \kappa(\mathfrak{p})$ equals zero. Then V is generated over $A[G]$ by a single element.*

Proof. Let x be a generator of $V^{(n)}$, and let $\tilde{x} \in V$ be a lift of x . If V' is the $A[G]$ -submodule of V generated by \tilde{x} , then $(V/V')^{(n)} = 0$. Since any non-generic quotient of $V \otimes \kappa(\mathfrak{p})$ equals zero, $(V/V') \otimes \kappa(\mathfrak{p}) = 0$ for all \mathfrak{p} . Since V/V' is admissible, we can apply Lemma 2.18 over the local rings $A_{\mathfrak{p}}$ to conclude V/V' is finitely generated, then apply ordinary Nakayama to conclude it is zero. \square

Thus every co-Whittaker module is admissible, Whittaker type, G -finite (in fact G -cyclic), and Schur, so satisfies the hypotheses of Theorem 3.5, below. Moreover, every admissible Whittaker type representation contains a canonical co-Whittaker submodule:

Proposition 2.30. *Let V in $\text{Rep}_A(G)$ be admissible of Whittaker type. Then the sub- $A[G]$ -module*

$$T := \ker(V \rightarrow \prod_{\{U \subset V: (V/U)^{(n)}=0\}} V/U)$$

is co-Whittaker.

Proof. $(V/T)^{(n)} = 0$ so T is Whittaker type. Since V is admissible so is T . Let \mathfrak{p} be a prime ideal and let $\overline{T} := T \otimes \kappa(\mathfrak{p})$. We show that $\cos(\overline{T})$ is absolutely irreducible and generic. By its definition, $\cos(\overline{T}) = \bigoplus_j W_j$ with W_j an irreducible $\kappa(\mathfrak{p})[G]$ -module. Since the map $\overline{T} \rightarrow \bigoplus_j W_j$ is a surjection and $(-)^{(n)}$ is exact and additive, the map $(\overline{T})^{(n)} \rightarrow \bigoplus_j W_j^{(n)}$ is also a surjection. Hence $\dim_{\kappa(\mathfrak{p})}(\bigoplus_j W_j^{(n)}) \leq \dim_{\kappa(\mathfrak{p})}(\overline{T}^{(n)})$. Since T is Whittaker type and $\overline{T}^{(n)} = \overline{T^{(n)}}$ is nonzero, there can only be one j such that $W_j^{(n)}$ is potentially nonzero. On the other hand, suppose some $W_j^{(n)}$ were zero, then W_j is a quotient appearing in the target of the map

$$\overline{V} \rightarrow \prod_{\{U \subset \overline{V}: (\overline{V}/U)^{(n)}=0\}} \overline{V}/U,$$

hence as a quotient of \overline{T} it would have to be zero, a contradiction. Hence precisely one W_j is nonzero. Now applying [EH12, 6.3.4] with A being $\kappa(\mathfrak{p})$ and V being $\cos(\overline{T})$, we have that $\text{End}_G(\cos(\overline{T})) \cong \kappa(\mathfrak{p})$ hence absolutely irreducible. It also shows that $\cos(\overline{T})^{(n)} = W_j^{(n)} \neq 0$. Hence $\overline{T}^{(n)} = \cos(\overline{T})^{(n)}$. By Lemma 2.29, \overline{T} is $\kappa(\mathfrak{p})[G]$ -cyclic; since it is admissible, it is finite length by Lemma 2.20. \square

2.6. The Integral Bernstein Center. If A is a Noetherian $W(k)$ -algebra and V is an $A[G]$ -module, then in particular V is a $W(k)[G]$ -module, so we use the Bernstein decomposition of $\text{Rep}_{W(k)}(G)$ to study V .

Let \mathfrak{W} be the $W(k)[G]$ -module $\text{c-Ind}_{N_n}^{G_n} \psi$. If e is a primitive idempotent of \mathcal{Z} , the representation $e\mathfrak{W}$ lies in the block $e\text{Rep}_{W(k)}(G)$, and we may view it as an object in the category $\text{Rep}_{e\mathcal{Z}}(G)$. With respect to extending scalars from $e\mathcal{Z}$ to A , the module $e\mathfrak{W}$ is “universal” in the following sense:

Proposition 2.31 ([Hel12b] Thm 6.3). *Let A be a Noetherian $e\mathcal{Z}$ -algebra. Then $e\mathfrak{W} \otimes_{e\mathcal{Z}} A$ is a co-Whittaker $A[G]$ -module. Conversely, if V is a primitive co-Whittaker $A[G]$ module in the block $e\text{Rep}_{W(k)}(G)$, and A is an $e\mathcal{Z}$ -algebra via*

$f_V : e\mathcal{Z} \rightarrow A$, then there is a surjection $\alpha : \mathfrak{W} \otimes_{A, f_V} A \rightarrow V$ such that $\alpha^{(n)} : (\mathfrak{W} \otimes_{A, f_V} A)^{(n)} \rightarrow V^{(n)}$ is an isomorphism.

If we assume A has connected spectrum (i.e. no nontrivial idempotents), then the map $f_V : \mathcal{Z} \rightarrow A$ would factor through a map $e\mathcal{Z} \rightarrow A$ for some primitive idempotent e , hence:

Corollary 2.32. *If A is a connected Noetherian $W(k)$ -algebra and V is co-Whittaker, then V must be primitive for some primitive idempotent e .*

Remark 2.33. *Theorems 1.1, 1.2, and 1.4 remain true if the hypothesis that V is primitive is replaced with the hypothesis that A is connected.*

3. ZETA INTEGRALS

In this section we use the representation theory of Section 2 to define zeta integrals and investigate their properties.

3.1. Definition of the Zeta Integrals. We first propose a definition of the zeta integral which is the analog of that in [JPSS79], and then check that the definition makes sense.

Definition 3.1. *For $W \in \mathcal{W}(V, \psi)$ and $0 \leq j \leq n-2$, let X be an indeterminate and define*

$$Z(W, X; j) = \sum_{m \in \mathbb{Z}} (q^{n-1} X)^m \int_{x \in F^j} \int_{a \in U_F} W \left[\begin{pmatrix} \varpi^m a & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix} \right] d^\times a dx,$$

and $Z(W, X) = Z(W, X; 0)$

We first show that $Z(W, X; 0)$ defines an element of $A[[X]][X^{-1}]$.

Lemma 3.2. *Let W be any element of $\text{Ind}_{N_n}^G \psi$. Then there exists an integer $N < 0$ such that $W(\begin{smallmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{smallmatrix})$ is zero for $v_F(a) < N$. Moreover if W is compactly supported modulo N_n , then there exists an integer $L > 0$ such that $W(\begin{smallmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{smallmatrix})$ is zero for $v_F(a) > L$*

Proof. There is an integer j such that $\begin{pmatrix} 1 & \mathfrak{p}^j & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$ stabilizes W . For x in \mathfrak{p}^j , we have

$$W \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{smallmatrix} \right) = W \left(\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \right) = \psi \left(\begin{smallmatrix} 1 & ax & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{smallmatrix} \right) W \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{smallmatrix} \right)$$

Whenever $v_F(a)$ is negative enough that ax lands outside of $\ker \psi = \mathfrak{p}$, we get that $\psi \left(\begin{smallmatrix} 1 & ax & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-2} \end{smallmatrix} \right)$ is a nontrivial p -power root of unity ζ in $W(k)$, hence $1 - \zeta$ is the lift of something nonzero in the residue field k , and defines an element of $W(k)^\times$. This shows that $W(\begin{smallmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_{n-1} \end{smallmatrix}) = 0$. \square

Just as in [JPSS79], the next two lemmas show that $Z(W, X; j)$ defines an element of $A[[X]][X^{-1}]$ when $0 < j < n-2$ by reducing it to the case of $Z(W, X; 0)$.

Lemma 3.3 ([JPSS79] Lemma 4.1.5). *Let H be a function on G , locally fixed under right translation by G , and satisfying $H(ng) = \psi(n)H(g)$ for $g \in G$, $n \in N_n$. Then the support of the function on F^j given by*

$$x \mapsto H \left[\begin{pmatrix} a & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-j-1} \end{pmatrix} \right]$$

is contained in a compact set independent of $a \in F^\times$.

Corollary 3.4. *If ρ denotes right translation $(\rho(g)\phi)(x) = \phi(xg)$, and U is the unipotent radical of the standard parabolic subgroup of type $(1, n-1)$, then there is a finite set of elements u_1, \dots, u_r of U such that*

$$Z(W, X; j) = \sum_{i=1}^r Z(\rho({}^t u_i)W, X; 0)$$

for any $W \in \text{Ind}_{N_n}^G \psi$.

In [JPSS79], the zeta integrals form elements of the field $\mathbb{C}((X))$ and it is proved that in fact they are elements of the subfield $\mathbb{C}(X)$ of rational functions. Whereas $\mathbb{C}((X))$ (resp. $\mathbb{C}(X)$) is the fraction field of the domain $\mathbb{C}[[X]]$ (resp. $\mathbb{C}[X, X^{-1}]$), our rings $A[[X]][X^{-1}]$ and $A[X, X^{-1}]$ are not in general domains. The first main result of this paper is determining the appropriate fraction ring of $A[X, X^{-1}]$ in which the zeta integrals $Z(W, X; j)$ live:

Theorem 3.5. *Suppose A is a Noetherian $W(k)$ -algebra. Let S be the multiplicative subset of $A[X, X^{-1}]$ consisting of polynomials whose first and last coefficients are units. Then if V is admissible, Whittaker type, and G -finite, then $Z(W, X; j)$ lies in $S^{-1}A[X, X^{-1}]$ for all W in $\mathcal{W}(V, \psi)$ for $0 \leq j \leq n-2$.*

In particular, the result holds if V is co-Whittaker, as in Theorem 1.1. The proof of Theorem 3.5 will occupy the remainder of this section. The key idea is that the zeta integrals $Z(W, X)$ are completely determined by the values $W\left(\begin{smallmatrix} a & 0 \\ 0 & I_{n-1} \end{smallmatrix}\right)$ for $a \in F^\times$, and as W ranges over $\mathcal{W}(V, \psi)$, the set of these values is equivalent to the data of the P_2 -representation $(\Phi^-)^{n-2}\mathcal{K}$. Determining the rationality of $Z(W, X)$ will then reduce to a finiteness result for the quotient $\mathcal{K}^{(n-1)}$, or more generally for $V^{(n-1)}$.

3.2. Proof of Rationality. Denote by τ the right translation representation of G_1 on $\mathcal{K}^{(n-1)}$. Let B be the commutative A -subalgebra of $\text{End}_A(\mathcal{K}^{(n-1)})$ generated by $\tau(\varpi)$ and $\tau(\varpi^{-1})$, where ϖ is a uniformizer of F . It follows from Corollary 2.25 that $\mathcal{K}_{\mathfrak{p}}^{(n-1)}$ is finitely generated over $A_{\mathfrak{p}}$. For every \mathfrak{p} of $\text{Spec } A$, the inclusion $B_{\mathfrak{p}} \subset \text{End}(\mathcal{K}^{(n-1)})_{\mathfrak{p}} \hookrightarrow \text{End}(\mathcal{K}_{\mathfrak{p}}^{(n-1)})$, shows $B_{\mathfrak{p}}$ is finitely generated as an $A_{\mathfrak{p}}$ -module.

Lemma 3.6. *B is finitely generated as an A -module.*

Proof. B is the image of the map $A[X, X^{-1}] \rightarrow \text{End}_A(\mathcal{K}^{(n-1)})$ sending X to $\tau(\varpi)$. $B_{\mathfrak{p}}$ is the image of the localized map $A_{\mathfrak{p}}[X, X^{-1}] \rightarrow (\text{End}_A(\mathcal{K}^{(n-1)}))_{\mathfrak{p}}$, which is finitely generated. Thus for every \mathfrak{p} , $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials $s_{\mathfrak{p}}(X)$, $t_{\mathfrak{p}}(X)$ with coefficients in $A_{\mathfrak{p}}$. Since $s_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$ have finitely many coefficients there exists a global section $f_{\mathfrak{p}} \notin \mathfrak{p}$ such that $s_{\mathfrak{p}}(X)$, $t_{\mathfrak{p}}(X)$ lie in $A_{f_{\mathfrak{p}}}[X]$. The open subsets $D(f_{\mathfrak{p}})$ cover $\text{Spec } A$ and we can take a finite subset $\{f_1, \dots, f_n\} \subset \{f_{\mathfrak{p}}\}$ such that $(f_i) = 1$. Since $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials over A_{f_i} , we have that B_{f_i} is finitely generated over A_{f_i} for each i . It follows that B is finitely generated over A . \square

Since B is finitely generated over A , $\tau(\varpi)$ and $\tau(\varpi^{-1})$ satisfy monic polynomials $c_0 + c_1X + \dots + c_{r-1}X^{r-1} + X^r$ and $b_0 + b_1X + \dots + b_{s-1}X^{s-1} + X^s$ respectively.

The degrees r and s are nonzero because $\tau(\varpi)$ and $\tau(\varpi^{-1})$ are units in B . Adding these together we have

$$0 = \tau(\varpi)^{-s} + b_{s-1}\tau(\varpi)^{-s+1} + \cdots + b_0 + c_0 + \cdots c_{r-1}\tau(\varpi)^{r-1} + \tau(\varpi)^r,$$

hence $\tau(\varpi)$ satisfies a Laurent polynomial whose first and last coefficients are units.

The final ingredient in proving rationality is the following transformation property.

Lemma 3.7. $Z(\varpi^n W, X) = X^{-n} Z(W, X)$ for any $W \in \mathcal{W}(V, \psi)$, and any integer n .

Proof of Lemma. Denote by b_m the coefficient $\int_{U_F} W(\varpi^m u) d^\times u$. Then $Z(\varpi^n W, X)$ is $\sum_{m \in \mathbb{Z}} X^m b_{m+n}$, which can be rewritten $X^{-n} Z(W, X)$. \square

Deducing Theorem 3.5. The representation $\mathcal{K}^{(n-1)}$ is formed by restricting the right translation representation on $(\Phi^-)^{n-2} \mathcal{K}$ from P_2 to G_1 , then taking the quotient by the G_1 -stable submodule $(\Phi^-)^{n-2} \mathcal{K}(U_2, \mathbf{1})$. By Corollary 2.12, the right translation representation on $(\Phi^-)^{n-2} \mathcal{K}$ is given by translations of the *restricted* Kirillov functions $W|_{\begin{pmatrix} x & 0 \\ 0 & I \end{pmatrix}}$, denoted $W(x)$ for short. As an endomorphism of the quotient module $\mathcal{K}^{(n-1)}$, $\tau(\varpi)$ satisfies a polynomial $X^n - a_{n-1}X^{n-1} - \cdots - a_1X - a_0$ (in fact we can take a_0 to be -1). Hence for any restricted Kirillov function $W(x)$ we have

$$\varpi^n W(x) = \sum_{i=0}^{n-1} a_i \varpi^i W(x) + W_1(x),$$

for some element W_1 of $((\Phi^-)^{n-2} \mathcal{K})(U_2, \mathbf{1})$. Therefore we get a relation

$$Z(\varpi^n W, X) = \sum_{i=0}^{n-1} a_i Z(\varpi^i W, X) + Z_1(X)$$

with $Z_1(X)$ being a Laurent polynomial by Lemma 3.2. Using Lemma 3.7, then multiplying through by X^n and rearranging we get $Z(W, X)(1 - \sum_{i=0}^{n-1} a_i X^{n-i}) = X^n Z_1(X)$ which completes the proof since a_0 is a unit. \square

4. FUNCTIONAL EQUATION AND GAMMA FACTOR

4.1. Contragredient Whittaker Functions. There is an analogue of the contragredient which is reflected on the level of Whittaker functions by a transform $\widetilde{(-)}$; the functional equation will relate the zeta integral of W to that of its transform. We will need the following two matrices:

$$w = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ & & \vdots & \\ (-1)^{n-1} & \cdots & 0 & 0 \end{pmatrix}, \quad w' = \begin{pmatrix} (-1)^n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & (-1)^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (-1)^0 & \cdots & 0 \end{pmatrix}$$

For any element W of $\text{Ind}_{N_n}^G \psi$, define the transform \widetilde{W} of W as $\widetilde{W}(g) := W(wg^t)$, where $g^t := {}^t g^{-1}$.

Observation 4.1. If V is of Whittaker type, then for $v \in V$, \widetilde{W}_v is an element of $\text{Ind}_N^G \psi$ because

$$\widetilde{W}(ng) = W(w(ng)^t) = W(wn^t w^{-1} w g^t) = \psi(wn^t w^{-1}) W(wg^t) = \psi(n) \widetilde{W}(g).$$

Thus $Z(\widetilde{w'W}, X; j)$ lands in $A[[X]][X^{-1}]$ by Lemma 3.2. In this section we state the second main result and recover the rationality properties of Section 2.2 for $Z(\widetilde{w'W}, X; j)$. The second main result is as follows:

Theorem 4.2. *Suppose A is a Noetherian $W(k)$ -algebra, and suppose V in $\text{Rep}_A(G)$ is co-Whittaker and primitive. Let S denote the multiplicative subset of Theorem 3.5. Then there exists a unique element $\gamma(V, X, \psi)$ of $S^{-1}A[X, X^{-1}]$ such that for any $W \in \mathcal{W}(V, \psi)$,*

$$Z(W, X; j)\gamma(V, X, \psi) = Z(\widetilde{w'W}, \frac{1}{q^n X}; n - 2 - j)$$

for $0 \leq j \leq n - 2$.

The proof of Theorem 4.2 is in Section 5. We now verify that $Z(\widetilde{w'W}, \frac{1}{q^n X}; j)$ always lives in $S^{-1}A[X, X^{-1}]$.

Proposition 4.3. *Suppose V in $\text{Rep}_A(G)$ is admissible, Whittaker type, G -finite, Schur, and primitive. Let V^ι denote the smooth $A[G]$ -module whose underlying A -module is V and whose G -action is given by $g \cdot v = g^\iota v$. Then V^ι is also admissible, Whittaker type, G -finite, Schur, and primitive.*

Proof. Consider the map $\text{Hom}_{N_n}(V, \psi) \rightarrow \text{Hom}_{N_n}(V^\iota, \psi)$ given by $\lambda \mapsto \tilde{\lambda}$, where $\tilde{\lambda} : x \mapsto \lambda(wx)$. We have $\tilde{\lambda}(n \cdot v) = \lambda(w n^\iota w^{-1} w v) = \psi(n) \tilde{\lambda}(v)$, which shows $\tilde{\lambda}$ indeed defines an element of $\text{Hom}_{N_n}(V^\iota, \psi)$. Since $w^2 = (-1)^{n-1} I_n$, it is an isomorphism of A -modules. Admissibility, G -finiteness, and Schur-ness all hold for V^ι since $g \mapsto g^\iota$ is a topological automorphism of the group G . Since V is Schur, A must be connected, hence V must be primitive since it is Schur. \square

In particular, $(V^\iota)^{(n)} = V^\iota / V^\iota(N_n, \psi)$ is free of rank one and we may define $(\widetilde{W})_v(g) = \tilde{\lambda}(g^\iota v)$ and take $\mathcal{W}(V^\iota, \psi)$ to be the A -module $\{(\widetilde{W})_v : v \in V^\iota\}$ as before. Note that this is precisely the same as $\{(\widetilde{W}_v) : v \in V\}$. We record this simple observation as a Lemma:

Lemma 4.4. *If λ is a generator of $\text{Hom}_{N_n}(V, \psi)$ then $\tilde{\lambda} : x \mapsto \lambda(wx)$ is a generator of $\text{Hom}_{N_n}(V^\iota, \psi)$ and defines $\mathcal{W}(V^\iota, \psi)$. There is an isomorphism of G -modules $\mathcal{W}(V, \psi) \rightarrow \mathcal{W}(V^\iota, \psi)$ sending W to \widetilde{W} .*

Thus all the hypotheses for the results of the previous sections, in particular Theorem 3.5, apply to V^ι whenever they apply to V , so we get $Z(\widetilde{w'W}, X; j)$ is in $S^{-1}A[X, X^{-1}]$. Now we can make the substitution $\frac{1}{q^n X}$ for X in the ratio of polynomials $Z(\widetilde{w'W}, X; j)$ to get $Z(\widetilde{w'W}, \frac{1}{q^n X}; j)$. It will again be in $S^{-1}A[X, X^{-1}]$ because this process swaps the first and last coefficients in the denominator (and q is a unit in A since q is relatively prime to ℓ).

4.2. Zeta Integrals and Tensor Products. The goal of this subsection is to check that the formation of zeta integrals commutes with change of base ring A . For any $f : A \rightarrow B$, denote by $\psi_A \otimes B$ the free rank one B -module with action given by the character $f \circ \psi$. The group action on $V \otimes_A B$ is given by acting in the first factor. Let i denote the map $V \rightarrow V \otimes_A B$. Proposition 2.1 (6), gives the following lemma.

Lemma 4.5. (1) *If V is of Whittaker type, so is $V \otimes_A B$.*

- (2) Let λ generate $\text{Hom}_{A[N]}(V, \psi)$ as an A -module. Then $\lambda \otimes \text{id}$ is a generator of $\text{Hom}_{B[N]}(V \otimes B, \psi \otimes B)$.
- (3) Let $W_{v \otimes b}(g) := (f \circ \lambda)(gv) \otimes b$ define elements of $\mathcal{W}(V \otimes B, \psi \otimes B)$. Then $f \circ W_v = W_{i(v)}$ for any $v \in V$.

From the definition of integration given in §1.1, it follows that if Φ_k is the characteristic function of some H_k , then $\int (f \circ \Phi_k) d(f \circ \mu^\times) = (f \circ \mu^\times)(H_k) = f(\int \Phi_k d(\mu^\times))$. It follows from the definitions that $(f \circ \widetilde{W})(x) = \widetilde{f \circ W}(x)$.

Corollary 4.6. *Let F denote the map of formal Laurent series rings $A[[X]][X^{-1}] \rightarrow B[[X]][X^{-1}]$ induced by f , then we have*

$$(1) \quad F(Z(W_v, X; j)) = Z(f \circ W, X; j) = Z(W_{i(v)}, X; j)$$

$$(2) \quad F\left(Z(\widetilde{w'W}, X; j)\right) = Z(f \circ \widetilde{w'W}, X; j) = Z(\widetilde{w'(f \circ W)}, X; j)$$

for any W in $\mathcal{W}(V, \psi)$, and for $0 \leq j \leq n-2$.

The next proposition follows from the linearity of the zeta integrals and the transform $(-)$.

Proposition 4.7. *Suppose there is an element $\gamma(V, X, \psi)$ in $A[[X]][X^{-1}]$ satisfying a functional equation as in Theorem 4.2 for all $W_v \in \mathcal{W}(V, \psi)$. Then the element $F(\gamma(V, X, \psi)) \in B[[X]][X^{-1}]$ satisfies the functional equation for all $W \in \mathcal{W}(V \otimes B, \psi \otimes B)$.*

4.3. Construction of the Gamma Factor. We define the gamma factor to be what it must in order to satisfy the functional equation of Theorem 4.2 for a single, particularly simple Whittaker function W_0 . We seek a W_0 such that $Z(W, X; 0)$ is a unit in $S^{-1}A[X, X^{-1}]$.

By Proposition 2.8 and Lemma 2.10, we have that $\text{c-Ind}_{U_2^{P_2}} \psi \subset (\Phi^-)^{n-2} \mathcal{K}$. Since $\text{c-Ind}_{U_2^{P_2}} \psi$ is isomorphic to $C_c^\infty(F^\times, A)$ via restriction to G_1 (recall that $C_c^\infty(F^\times, A)$ denotes the locally constant compactly supported functions $F^\times \rightarrow A$), we find the following:

Proposition 4.8. *Suppose V in $\text{Rep}_A(G)$ is of Whittaker type. Then the characteristic function of U_F^1 is realized as a restricted Whittaker function $W_0(\begin{smallmatrix} g & 0 \\ 0 & I_{n-1} \end{smallmatrix})$ for some W_0 in $\mathcal{W}(V, \psi)$.*

From now on, the symbol W_0 will denote a choice of element in $\mathcal{W}(V, \psi)$ whose restriction to $(\begin{smallmatrix} g & 0 \\ 0 & I_{n-1} \end{smallmatrix})$ is the characteristic function of U_F^1 . Then $Z(W_0, X)$ is $\int_{U_F} W_1(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) d^\times a = \mu^\times(U_F^1) = 1$. Since we want our gamma factor to satisfy the functional equation for W_0 , we are left with no choice:

Definition 4.9 (The Gamma Factor). *Let A be any Noetherian $W(k)$ -algebra and suppose V in $\text{Rep}_A(G)$ is of Whittaker type. We define the gamma factor of V with respect to ψ to be the element of $A[[X]][X^{-1}]$ given by $\gamma(V, X, \psi) := Z(\widetilde{w'W_0}, \frac{1}{q^n X}; n-2)$.*

When V is co-Whittaker and primitive the uniqueness of this gamma factor will follow from the functional equation: if γ and γ' both satisfy the functional equation for all Whittaker functions, then $\gamma = Z(\widetilde{w'W_0}, \frac{1}{q^n X}, n-2) = \gamma'$. In particular for such representations our construction of the gamma factor does not depend on the choice of W_0 .

4.4. Functional Equation for Characteristic Zero Points. If the residue field $\kappa(\mathfrak{p})$ of \mathfrak{p} has characteristic zero, the reduction modulo \mathfrak{p} of $Z(W, X; j)$ forms an element of $\overline{\kappa(\mathfrak{p})}(X)$. As $\overline{\kappa(\mathfrak{p})}$ is an uncountable algebraically closed field of characteristic zero, we may fix an embedding $\mathbb{C} \hookrightarrow \overline{\kappa(\mathfrak{p})}$. The proof of [JPSS83, Thm 2.7(iii)(2)] (which occurs in [JPSS83, §2.11]) carries over verbatim to the setting where π and π' are admissible, Whittaker type, G -finite representations over any field containing \mathbb{C} , hence for representations over $\kappa(\mathfrak{p})$. Thus the reduction modulo \mathfrak{p} of $\Psi(W, X; j)$ is precisely the integral $\Psi(s, W; j)$ of [JPSS79, §4.1], after replacing the complex variable $q^{-(s+\frac{n-1}{2})}$ with the indeterminate X , and there exists a unique element, which we will call $\gamma_{\mathfrak{p}}(s, V \otimes \overline{\kappa(\mathfrak{p})}, \psi)$, in $\overline{\kappa(\mathfrak{p})}(q^{-s})$ such that for all $W \in \mathcal{W}(V_{\mathfrak{p}} \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$ and for all $j \geq 0$,

$$\Psi(1-s, \widetilde{w'W}; n-2-j) = \gamma_{\mathfrak{p}}(s, V \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}}) \Psi(s, W, j).$$

The change of variable $s \mapsto 1-s$ can be re-written as $-(s+\frac{n-1}{2}) \mapsto (s+\frac{n-1}{2})-n$, so writing the functional equation in terms of X we have shown the following Lemma:

Lemma 4.10. *Suppose V is admissible of Whittaker type, and G -finite. For each prime \mathfrak{p} of A with residue characteristic zero, there exists a unique element $\gamma_{\mathfrak{p}}(V \otimes \overline{\kappa(\mathfrak{p})}, X, \psi_{\mathfrak{p}})$ in $\overline{\kappa(\mathfrak{p})}(X)$ such that for all W in $\mathcal{W}(V \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$ and for $0 \leq j \leq n-2$ we have*

$$Z(\widetilde{w'W}, \frac{1}{q^n X}; n-2-j) = \gamma_{\mathfrak{p}}(X, V \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}}) Z(W, X; j).$$

Moreover, $\gamma_{\mathfrak{p}}(V \otimes \overline{\kappa(\mathfrak{p})}, X, \psi_{\mathfrak{p}}) = \gamma(V, X, \psi) \pmod{\mathfrak{p}}$ by uniqueness in [JPSS79].

4.5. Proof of Functional Equation When A is Reduced and ℓ -torsion Free. In the case that A is reduced and ℓ -torsion free as a $W(k)$ -algebra, we get a slightly stronger result than that of Theorem 4.2.

Theorem 4.11. *If A is a Noetherian $W(k)$ -algebra and A is reduced and ℓ -torsion free, then the conclusion of Theorem 4.2 holds for any V in $\text{Rep}_A(G)$ which is G -finite, and admissible of Whittaker type.*

Proof. Let \mathfrak{p} be any characteristic zero prime, and let $f_{\mathfrak{p}} : A \rightarrow \overline{\kappa(\mathfrak{p})}$ be reduction modulo \mathfrak{p} . Corollary 4.6 and Lemma 4.10 tell us that

$$f_{\mathfrak{p}} \left(\gamma(V, X, \psi) Z(W, X) - Z(\widetilde{w'W}, \frac{1}{q^n X}; n-2) \right) = 0$$

for any W in $\mathcal{W}(V, \psi)$, not just W_0 . This shows that the difference

$$\gamma(V, X, \psi) Z(W, X) - Z(\widetilde{w'W}, \frac{1}{q^n X}; n-2)$$

is in the intersection of all characteristic zero primes of A . When A is reduced its zero divisors are the union of its minimal primes, so it is ℓ -torsion free if and only if all minimal primes have residue characteristic zero. Thus when A is reduced and ℓ -torsion free, the intersection of all characteristic zero primes of A equals zero, so the functional equation holds for any W in $\mathcal{W}(V, \psi)$.

We now prove uniqueness. If there were another element γ' satisfying the same property, it would satisfy the functional equation in $\overline{\kappa(\mathfrak{p})}$ for all $W_{i(v)}$ by reduction, so it satisfies the functional equation for all W in $\mathcal{W}(V \otimes \overline{\kappa(\mathfrak{p})}, \psi_{\mathfrak{p}})$. But uniqueness in Lemma 4.10 then shows $f_{\mathfrak{p}}(\gamma(V, X, \psi) - \gamma') = 0$ for all characteristic zero primes \mathfrak{p} of A . Again, this means $\gamma' = \gamma(V, X, \psi)$.

We get rationality by observing that whenever V is admissible of Whittaker type, it has a canonical co-Whittaker submodule T by Proposition 2.30, which is primitive if V is primitive. Since $\gamma(T, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$, we must have $\gamma(T, X, \psi) = \gamma(V, X, \psi)$ by the construction of the gamma factor. But $\gamma(T, X, \psi)$ is in $S^{-1}A[X, X^{-1}]$ by Theorem 3.5, which holds for primitive co-Whittaker modules. \square

5. UNIVERSAL GAMMA FACTORS

When V is primitive and co-Whittaker, we can remove the hypothesis that A is reduced and ℓ -torsion free by specializing the gamma factor for the universal co-Whittaker module $e\mathfrak{W}$.

Theorem 5.1 ([Hel12a] Thm 12.1). *Any block $e\mathcal{Z}$ of the Bernstein center of $\text{Rep}_{W(k)}(G)$ is a finitely generated (hence Noetherian), reduced, ℓ -torsion free $W(k)$ -algebra.*

By Proposition 2.31, $e\mathfrak{W}$ is co-Whittaker, and since it is clearly primitive, all the hypotheses of Proposition 4.11 are satisfied. Hence (Thm 4.11) there exists a unique gamma factor in $S^{-1}(e\mathcal{Z}[X, X^{-1}])$, which we will denote $\Gamma(e\mathfrak{W}, X, \psi)$, satisfying the functional equation for all W in $\mathcal{W}(e\mathfrak{W}, \psi)$.

Proof of Theorem 4.2. Since V is primitive and co-Whittaker, there is a (unique) primitive idempotent e of \mathcal{Z} and a ring homomorphism $f_V : e\mathcal{Z} \rightarrow \text{End}_G(V) \xrightarrow{\sim} A$, and a surjection of $A[G]$ -modules $e\mathfrak{W} \otimes_{f_V} A \rightarrow V$ preserving the top derivative, so that $f_V(\Gamma(e\mathfrak{W}, X, \psi)) = \gamma(e\mathfrak{W} \otimes_{f_V} A, X, \psi)$. Since $\Gamma(e\mathfrak{W}, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(e\mathfrak{W}, \psi)$, we can apply Proposition 4.7 again to conclude that $\gamma(e\mathfrak{W} \otimes A, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(e\mathfrak{W} \otimes A, \psi)$. Since $e\mathfrak{W} \otimes A$ has a surjection onto V preserving the top derivative, Lemma 2.6 tells us that $\mathcal{W}(V, \psi) = \mathcal{W}(e\mathfrak{W} \otimes A, \psi)$. The functional equation shows that Definition 4.9 gives a unique gamma factor, hence $\gamma(V, X, \psi) = \gamma(e\mathfrak{W} \otimes A, X, \psi)$; it satisfies the functional equation for all W in $\mathcal{W}(V, \psi)$. Note that since $\Gamma(e\mathfrak{W}, X, \psi)$ is in $S^{-1}(e\mathcal{Z}[X, X^{-1}])$, its image in f_V is in the corresponding fraction ring of $A[X, X^{-1}]$. This proves Theorem 4.2. \square

We can extend the uniqueness and rationality result to a larger class of representations, though with a weaker functional equation coming only from the co-Whittaker case:

Corollary 5.2. *Let V be admissible, primitive, of Whittaker type and let T be its canonical co-Whittaker submodule. Then there exists a unique gamma factor $\gamma(V, X, \psi)$ in $S^{-1}(A[X, X^{-1}])$ which equals $\gamma(T, X, \psi)$, and satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$.*

Proof. When V is admissible of Whittaker type it has a canonical co-Whittaker sub by Proposition 2.30. We have just shown that its gamma factor $\gamma(T, X, \psi)$ satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$. Applying Proposition 2.6 with $\alpha : T \rightarrow V$ being the inclusion map, we conclude that $\mathcal{W}(T, \psi) \subset \mathcal{W}(V, \psi)$. The coefficients of the series $Z(\widetilde{w'W_0}, \frac{1}{q^n X}; n-2)$ in Definition 4.9 are determined by G -translates of the Whittaker function W_0 , so occurs already in $\mathcal{W}(T, \psi)$, so by definition $\gamma(T, X, \psi) = \gamma(V, X, \psi)$. In particular $\gamma(V, X, \psi)$ lies in $S^{-1}A[X, X^{-1}]$ and satisfies the functional equation for all W in $\mathcal{W}(T, \psi)$. \square

We can make precise the sense in which we have created a universal gamma factor:

Corollary 5.3. *Suppose A is a Noetherian $W(k)$ -algebra, and suppose V is a co-Whittaker $A[G]$ -module in the subcategory $e\mathrm{Rep}_{W(k)}(G)$ of $\mathrm{Rep}_{W(k)}(G)$. Then there is a homomorphism $f_V : e\mathcal{Z} \rightarrow A$ and $f_V(\Gamma(e\mathfrak{W}, X, \psi))$ equals the unique $\gamma(V, X, \psi)$ satisfying a functional equation for all W in $\mathcal{W}(V, \psi)$.*

Again, we can broaden the class of representations at the cost of a more restrictive functional equation:

Theorem 5.4. *Suppose A is any Noetherian $W(k)$ -algebra, and suppose V is an admissible $A[G]$ -module of Whittaker type in the subcategory $e\mathrm{Rep}_{W(k)}(G)$. Then there is a homomorphism $f_V : e\mathcal{Z} \rightarrow A$ and the gamma factor of Corollary 5.2 equals $f_V(\Gamma(e\mathfrak{W}, X, \psi))$.*

Proof. We define f_V to be the homomorphism $e\mathcal{Z} \rightarrow \mathrm{End}_G(T) \xrightarrow{\sim} A$ where T is the canonical co-Whittaker submodule of Proposition 2.30. Since T lies in $e\mathrm{Rep}_{W(k)}(G)$, $e\mathfrak{W} \otimes_{f_V} A$ surjects onto T , and we have $f_V(\Gamma(e\mathfrak{W}, X, \psi)) = \gamma(T, X, \psi)$ (Prop 2.6), and since T injects into V (with top derivative preserved), again by Prop 2.6, we have

$$f_V(\Gamma(e\mathfrak{W}, X, \psi)) = \gamma(e\mathfrak{W} \otimes_{e\mathcal{Z}, f_V} A, X, \psi) = \gamma(T, X, \psi) = \gamma(V, X, \psi).$$

□

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